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The stability of an asymmetric zonal current in the atmosphere

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This paper considers the barotropic stability of an asymmetric zonal current on a rotating earth. The current is of hyperbolic tangent form in latitude. For this stability problem the neutral wave solutions are found and estimates of the growth rates for the unstable waves are obtained from the neutral solutions as in earlier investigations by the writer. It is again found that the beta effect which is due to the earth's curvature and rotation tends to stabilize the basic flow.

The stability of the basic flow is examined for a special case. For this case the current is centered at 35 degrees latitude, has a total shear of 30 m sec^{-1} and a half-width of 550 km. The most unstable waves are found to be wave-numbers 6 and 7 which amplify by a factor of e in 2·9 days. In addition, wave-numbers 5, 8 and 9 are also unstable. The stability of the symmetric jet is also examined for a comparable case. It is found that a wider band of wave-numbers is unstable. The most unstable wave is wave-number 8 which amplifies by a factor of e in 2·7 days.

In conclusion it is noted that these growth rates are slower than the amplification rates for the unstable waves associated with the baroclinic stability problem.

1. Introduction

The motivation for this investigation is the problem of the stability of the westerly winds in the atmosphere. These winds vary with latitude and height and are strongest at the tropopause. When latitudinal variations are considered this current has a jet-like appearance with the maximum winds occurring along a certain latitude which changes from day to day. For the mean winter circulation the maximum winds are near 30° N. (Mintz 1955). At present, a solution of the three-dimensional stability problem involving continuous variations of the basic flow with latitude and height is too complicated for mathematical treatment. For this reason most investigators have considered one of two approaches. The first is to allow the current to vary with height but not latitude. This formulation is known as the baroclinic stability problem (see Charney 1947; Kuo 1952). The other approach is to allow the current to vary with latitude but not height. This formulation is known as the barotropic stability problem (Kuo 1949). In both cases frictional forces are neglected.

It is indeed legitimate to ask what relevance these two simplified problems have to the more complicated motions that occur in the atmosphere. It is

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observed that disturbances with many properties of the simplified baroclinic waves amplify with time and feed energy into the basic current (see Phillips 1963). This type of instability appears to be the main mode of instability for the large-scale motions in the atmosphere. However, the atmosphere also appears to be barotropically unstable at times, particularly at 500 mb. For synoptic examples of barotropic instability at 500 mb see Kuo (1949) and Wiin-Nielsen (1961).

As in Lipps (1962) we are concerned with the barotropic stability problem. This problem has also been considered by Foote & Lin (1951) and Kuo (1949, 1951). These authors show that the barotropic basic current is stable if the gradient of absolute vorticity is of one sign throughout the fluid. It is also found in the above investigations that the effect of the earth's rotation is to reduce the instability of the basic current. In this respect the effect of the earth's rotation is similar to that of a density stratification in stabilizing a shearing basic current. This latter stability problem has been treated by Drazin (1958) and later investigators, e.g. Howard (1963).

Most of the previous studies on barotropic stability have considered the stability of a symmetric basic current with a jet-like appearance. These basic flows have regions with negative absolute vorticity gradient both to the north and to the south of the maximum winds, elsewhere the gradient is positive. There is no reason to think that this is the only type of barotropically unstable flow to occur in the atmosphere. An unstable jet may have a region of negative absolute vorticity gradient only to the north or to the south of the maximum winds. Furthermore, an instability may occur in a basic shearing current which does not have a jet-like appearance but which has different constant values for the velocity to the north and south of the region of strong shear.

In order to consider the stability characteristics of the latter types of flow we study the stability of an asymmetric velocity profile of hyperbolic tangent form in latitude. The results of the present investigation will be applied to the case of a particular zonal current. Then these results will be compared to the results obtained from the analysis of the symmetric jet (Lipps 1962, 1963).

2. The perturbation equations and boundary conditions

In this study the motion is assumed to be horizontal, non-divergent and barotropic. The basic flow consists of a fluid current streaming from west to east. As in Lipps (1962) we assume that the variation of the basic current occurs in a very narrow latitude belt, and that the basic flow is constant on either side of this latitude belt. It is therefore legitimate to approximate the spherical co-ordinates of the earth by Cartesian co-ordinates x, y and z directed toward the east, north and vertical, respectively (see Long 1960). The corresponding velocities are u, v and w. The basic flow is of the form $U = U(y)$. For this velocity profile we take

$$U = U_0 \tanh y/L + U_1, \tag{2.1}$$

where U_0, L and U_1 are arbitrary constants which will be specified in any particular case. Since the motion is non-divergent and horizontal, we may define a stream function for the perturbed flow,

$$u = -\partial\psi/\partial y, \quad v = \partial\psi/\partial x.$$

The perturbations must satisfy the two-dimensional vorticity equation as in Lipps (1962). This equation takes the form

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\nabla^2\psi+(\beta_0-U'')\frac{\partial\psi}{\partial x}=0. \tag{2.2}$$

In this equation a prime denotes a differentiation with respect to y and $\beta_0=(d/dy)\,2\omega$ where ω is the vertical component of the earth's rotation. As in Lipps (1962) we may consider β_0 to be a constant since the variation of the basic flow occurs over a narrow latitude belt.

This approximation in which the spherical co-ordinates on the earth are replaced by Cartesian co-ordinates and β_0 is considered constant is known as the beta-plane approximation in meteorological literature (see Rossby and collaborators 1939; Lipps 1963; Phillips 1963). A physical interpretation of this approximation is that all the effects of the earth's curvature are unimportant except for the variation with latitude of the vertical component of rotation, and furthermore that this variation may be considered as constant.

Now we set $\psi(x,y,t)=e^{i\alpha(x-ct)}\phi(y)$ so that (2.2) becomes

$$\phi''-\alpha^2\phi+\frac{\beta_0-U''}{U-c}\phi=0, \tag{2.3}$$

where α is the wave-number and c is the phase velocity which may be complex, i.e.

$$c=c_r+ic_i.$$

The boundary conditions on ϕ are that $\phi\to 0$ as $y\to\pm\infty$.

It is convenient to change to non-dimensional variables. For this purpose we define

$$\left.\begin{aligned}&x^*=x/L;\quad y^*=y/L;\quad t^*=t|U_0|/L;\quad k=\alpha L,\\ &c^*=\frac{c-U_1}{|U_0|};\quad \beta=\frac{\beta_0L^2}{|U_0|};\quad U^*=\frac{U_0}{|U_0|}\tanh y^*;\quad \phi^*=\frac{\phi}{|U_0|L}.\end{aligned}\right\} \tag{2.4}$$

Dropping the asterisks equations (2.3) becomes

$$\phi''-k^2\phi+\frac{\beta-U''}{U-c}\phi=0. \tag{2.5}$$

The boundary conditions on ϕ remain the same, namely $\phi\to 0$ as $y\to\pm\infty$.

We now show that the solution to (2.5) has a remarkable property. In the non-dimensionalized form of U we note that there is either a plus or minus sign in front of $\tanh y$ depending on whether U_0 is positive or negative. If we set $\eta=-y$, the form of (2.5) remains the same with y replaced by η. However, the form of U given by (2.4) changes to

$$U=-\frac{U_0}{|U_0|}\tanh\eta.$$

Thus the only effect of a reflection of the y-axis is to change the sign of the basic velocity. Hence if we have a solution to (2.5) for a particular k^2 and a positive U_0

given by $\phi_+(y)$ and the phase velocity c_+, the solution for the same k^2 and U_0 of opposite sign will be

$$\phi_-(y) = \phi_+(-y), \quad c_- = c_+, \tag{2.6}$$

where ϕ_- and c_- refer to the solution for the negative U_0. This result shows that once we find the solution to the stability problem for a positive U_0 we also have the solution when U_0 takes the opposite sign. In either case the stability of the basic flow is the same since the phase velocities are identical.

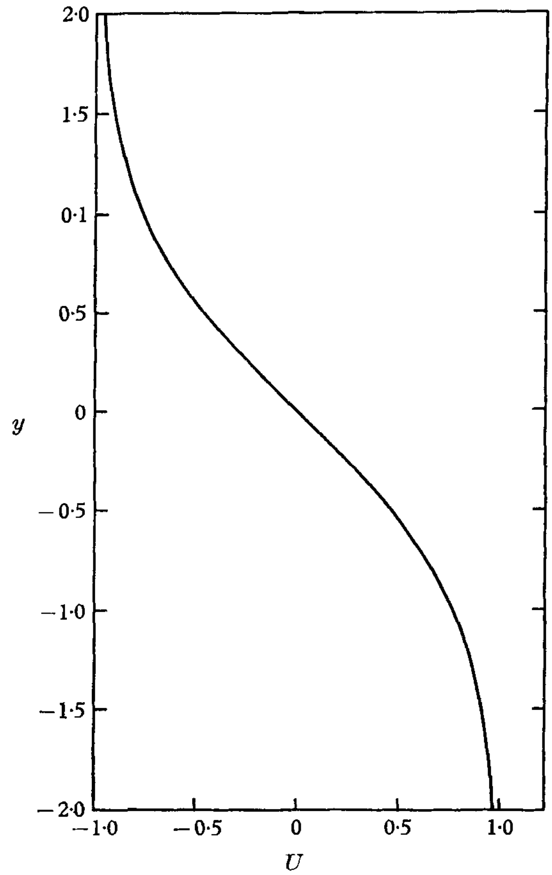

FIGURE 1. $U = -\tanh y$.

Thus as far as the stability analysis is concerned, which sign we take for the basic flow is arbitrary. Physically, if we consider $U = -\tanh y$, we are considering a basic flow which has a negative absolute vorticity gradient north of the region of maximum winds. Likewise, if we consider $U = \tanh y$, we are considering a basic flow which has a negative absolute vorticity gradient south of the region of maximum winds. There is some evidence to suggest that the former case occurs more frequently than the latter case (see Kuo 1949). Therefore we choose to take $U = -\tanh y$ for the basic flow pattern.

The non-dimensional basic velocity is shown in figure 1. In figure 2 we have plotted some typical profiles of absolute vorticity. In this figure the absolute vorticity is given by

$$\zeta - \zeta_0 = \beta y - U',$$

where ζ_0 is the absolute vorticity at $y = 0$ due to the rotation of the earth. It is to be noted that for $\beta > 4/3^{\frac{3}{2}}$, the absolute vorticity profile is monotonic. Therefore the flow is stable for these values of β.

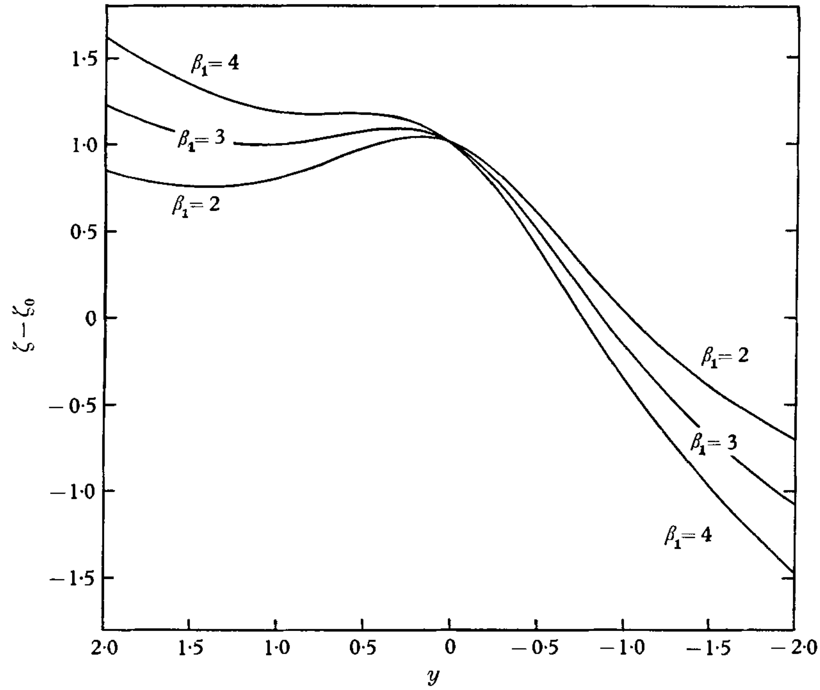

FIGURE 2. The absolute vorticity profiles associated with $U = -\tanh y$ for representative values of $\beta_1 = 3^{\frac{3}{2}}\beta$.

3. The neutral solutions

We now consider the neutral wave solutions to (2.5). The neutral waves that are associated with the stability problem are of most interest and will be the major concern of this section. This discussion is similar to that of Lipps (1962).

Certain neutral waves are relevant to the stability problem because unstable waves exist for adjacent wave-numbers. From these neutral wave solutions it is possible to calculate $\partial c/\partial k^2$ and make an estimate of the amplification rates for the unstable waves. This question will be considered in the next section.

For these neutral waves it can be shown that $U = c$ at $y = y_c$ where the gradient of absolute vorticity given by $\beta - U''$ vanishes. This result follows since the basic velocity is a monotonic function of y (see Foote & Lin 1951).

With this information we may obtain the values of c for these waves. At $y = y_c$ where $U = c$ we have $c = -\tanh y_c$. Thus setting $\beta - U'' = 0$ we find

$$\beta + 2(1-c^2)\,c = 0. \tag{3.1}$$

This equation gives the following three roots for c

$$c_1 = (2/\sqrt{3})\cos\tfrac{1}{3}\theta, \quad c_2 = (2/\sqrt{3})\cos(\tfrac{1}{3}\theta + 120^\circ),$$
$$c_3 = (2/\sqrt{3})\cos(\tfrac{1}{3}\theta + 240^\circ), \tag{3.2}$$

where $$\theta = \cos^{-1}(\beta 3^{\frac{3}{2}}/4), \quad 0 \leqslant \theta \leqslant \tfrac{1}{2}\pi.$$

The first root gives a value for c greater than unity and is spurious since Kuo (1949) proves that no values of c exist which are greater than the maximum basic velocity. The other two roots give values of c which are of interest. These roots are real for $0 \leqslant \beta \leqslant 4/3^{\frac{3}{2}}$ but become complex for larger values of β. Since we know that the flow is stable for $\beta > 4/3^{\frac{3}{2}}$ it is evident that these two roots are no longer relevant for $\beta > 4/3^{\frac{3}{2}}$.

If we set c equal to either c_2 or c_3 equation (2.5) becomes

$$\phi'' + [2\operatorname{sech}^2 y + 2c\tanh y - 2c^2 - k^2]\,\phi = 0. \tag{3.3}$$

In this equation c is known. Therefore the problem is to find a value of k^2 for which a solution of (3.3) exists and to find the form of the solution. For this purpose it is convenient to change the independent variable from y to $z = \tanh y$. Then (3.3) becomes

$$(1-z^2)\,\phi_{zz} - 2z\phi_z + \left(2 - \frac{k^2 + 2c^2 - 2cz}{1-z^2}\right)\phi = 0, \tag{3.4}$$

where the subscript z denotes differentiation with respect to z. From this equation it is evident that ϕ has singularities at $z = \pm 1$. These singularities can be removed by a change of the dependent variable

$$\phi = (1+z)^{\frac{1}{2}\alpha_1}(1-z)^{\frac{1}{2}\alpha_2}\chi, \tag{3.5}$$

where $$\alpha_1^2 = k^2 + 2c + c^2, \quad \alpha_2^2 = k^2 - 2c + c^2.$$

With this transformation (3.4) becomes

$$\chi_{zz} - \left(\frac{1+\alpha_1}{1+z} + \frac{1+\alpha_2}{z-1}\right)\chi_z + \frac{\sigma + \tau z^2}{(z-1)^2\,(z+1)^2}\chi = 0, \tag{3.6}$$

where $$\sigma = (\alpha_1 - \alpha_2)^2/4 - (\alpha_1 + \alpha_2)/2 + 2 - 2c^2 - k^2,$$
$$\tau = (\alpha_1 + \alpha_2)(\alpha_1 + \alpha_2 + 2)/4 - 2.$$

A solution of the form $\chi = \text{const.}$ exists provided σ and τ vanish. It can be shown that such a solution does exist when

$$\alpha_1 = 1 + c, \quad \alpha_2 = 1 - c, \quad k^2 = 1 - c^2. \tag{3.7}$$

Then ϕ is of the form $$\phi = (1+z)^{\frac{1}{2}(1+c)}(1-z)^{\frac{1}{2}(1-c)}. \tag{3.8}$$

This method of solution closely parallels that of Drazin (1958).

The relation $k^2 = 1 - c^2$ defines a neutral curve in the (β, k^2)-plane since $c = c(\beta)$ as given by c_2 and c_3 in equation (3.2). This curve is shown in figure 3. For $\beta < 4/3^{\frac{3}{2}}$ two neutral waves exist for a given β. The waves with phase velocity c_2 are along the lower portion of the neutral curve in figure 3 and the waves with phase velocity c_3 are along the upper neutral curve. At $\beta = 4/3^{\frac{3}{2}}$ there is only one neutral wave with the phase velocity $c = -3^{-\frac{1}{2}}$. In the next section it will be shown that waves are amplified for wavelengths between those of the two neutral waves. Outside this wavelength band there are only stable waves. Some typical values of k^2 and c for the neutral waves are given in table 1.

In addition to these neutral waves which are relevant to the stability problem there are two other types of neutral waves. The first type we call the Rossby (1939)–Haurwitz (1940) waves. These waves move with phase velocities less than the minimum basic velocity. The second type of neutral waves are those which have a continuous spectrum of phase velocities such that $U = c$ for some point within the fluid. For a more complete discussion of both types of neutral waves see Lipps (1963).

$\beta 3^{\frac{3}{2}}$	k^2	c	$\partial c/\partial s$	$\partial c/\partial \beta$
0	1·000	0	$0 + 0{\cdot}318i$	$-0{\cdot}500 - 0i$
1	0·991	−0·097	$-0{\cdot}091 + 0{\cdot}313i$	$-0{\cdot}505 - 0{\cdot}031i$
2	0·960	−0·200	$-0{\cdot}190 + 0{\cdot}229i$	$-0{\cdot}525 - 0{\cdot}068i$
3	0·896	−0·322	$-0{\cdot}333 + 0{\cdot}260i$	$-0{\cdot}570 - 0{\cdot}122i$
4	0·667	−0·578	$-0{\cdot}866 + 0i$	$-0{\cdot}906 - 0{\cdot}340i$
3	0·361	−0·799	$-2{\cdot}96 - 1{\cdot}19i$	$-2{\cdot}04 - 1{\cdot}04i$
2	0·217	−0·885	$-7{\cdot}02 - 2{\cdot}56i$	$-4{\cdot}24 - 1{\cdot}68i$
1	0·101	−0·948	$-19{\cdot}04 - 3{\cdot}65i$	$-10{\cdot}6 - 2{\cdot}04i$
0	0	−1·000	$-\infty - 3{\cdot}14i$	$-\infty - 1{\cdot}57i$

TABLE 1. Values calculated along the neutral curve in figure 3

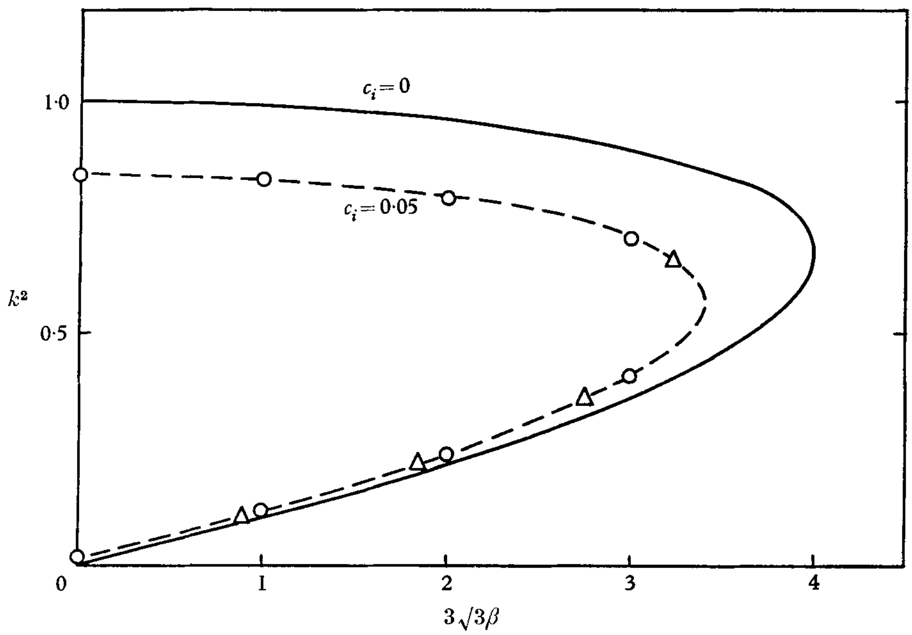

FIGURE 3. Stability of $U = -\tanh y$. The solid line is the neutral curve and the dashed line is the estimated curve for $c_i = 0{\cdot}05$. The triangles represent estimates of $c_i = 0{\cdot}05$ from the values of $\partial c_i/\partial \beta$ along the neutral curve and the circles represent estimates of $c_i = 0{\cdot}05$ from the values of $\partial c_i/\partial s$ along the neutral curve.

4. Amplified waves

To find c near the neutral curve in figure 3 we may expand c in a Taylor series of the form

$$c = c_0 + \frac{\partial c}{\partial s} ds + \frac{\partial c}{\partial \beta} d\beta + \dots, \tag{4.1}$$

where $s = -k^2$ and c_0, $\partial c/\partial s$ and $\partial c/\partial \beta$ are evaluated at some point on the neutral curve. In the following $\partial c/\partial s$ and $\partial c/\partial \beta$ are calculated from the neutral solution and the higher derivatives are neglected. Thus for points close to the neutral curve we should find a good approximation for c.

To calculate $\partial c/\partial s$ and $\partial c/\partial \beta$ along the neutral curve we use the approach previously given by Lipps (1962). These expressions are given by

$$\frac{\partial c}{\partial s} = -\int_{-\infty}^{\infty} \phi^2\,dy \Big/ \int_{-\infty}^{\infty} \frac{\beta - U''}{(U-c)^2}\phi^2\,dy, \tag{4.2}$$

$$\frac{\partial c}{\partial \beta} = -\int_{-\infty}^{\infty} \frac{1}{U-c}\phi^2\,dy \Big/ \int_{-\infty}^{\infty} \frac{\beta - U''}{(U-c)^2}\phi^2\,dy, \tag{4.3}$$

where ϕ is the neutral solution given by (3.8).

Both of these expressions have a singularity in the denominator where $U = c_0$ so that we must integrate around this point in the complex y-plane. To decide whether to integrate above or below this point we use the criterion of Foote & Lin (1951). By considering the viscous solution in the limit of vanishing viscosity they find that the path of integration should be above the point if $dU/dy < 0$. A similar situation holds for the singularity in the numerator of (4.3).

In the previous paper (Lipps 1962) it was possible to evaluate the expressions (4.2) and (4.3) analytically after changing the variable to $z = \tanh y$. In the present case most of the values must be calculated numerically. An exception is the values of $\partial c/\partial s$ and $\partial c/\partial \beta$ at $\beta = 0$, $k^2 = 1$ which can be found analytically. The limiting values of $\partial c/\partial s$ and $\partial c/\partial \beta$ as $\beta \to 0$, $k^2 \to 0$ along the neutral curve can also be found analytically. This calculation is given in the Appendix. All the computed values of $\partial c/\partial s$ and $\partial c/\partial \beta$ are shown in table 1.

The estimated curve for $c_i = 0{\cdot}05$ shown in figure 3 is found from the values of $\partial c/\partial s$ and $\partial c/\partial \beta$ given in table 1. In this figure the triangles represent estimates of $c_i = 0{\cdot}05$ from the values of $\partial c_i/\partial \beta$ along the neutral curve and the circles represent estimates of $c_i = 0{\cdot}05$ from the values of $\partial c_i/\partial s$ along the neutral curve.

It is of interest to note the behaviour of the derivatives $\partial c/\partial s$ and $\partial c/\partial \beta$ in the limit as $k^2, \beta \to 0$. From table 1 we see that the imaginary parts of these derivatives have finite values in the limit while the real parts do not. Thus in figure 3 the $c_i = 0{\cdot}05$ curve remains a finite distance from the origin.† For the symmetric jet discussed in Lipps (1962) neither the real nor the imaginary parts of the corresponding derivatives $\partial c/\partial s$ and $\partial c/\partial x^{-1}$ remained finite in the limiting process. On the other hand, when a jet in a divergent, barotropic fluid (Lipps 1963) was considered, it was found that $\partial c/\partial s$ and $\partial c/\partial \beta$ had finite real values at the origin of the (β, k^2)-plane. Thus the behaviour of these derivatives at this point is rather critically determined by the details of the stability problem considered.

It is also of interest to note that for the present basic flow there is only one mode of unstable disturbances as determined by the above analysis. In Lipps (1962) there were two modes of unstable waves; namely, the symmetric disturbances and the antisymmetric disturbances. One could indeed ask whether the extremely simple neutral solutions found to equation (3.4) include all the neutral

† Drazin & Howard (1962) and Michalke (1964) have treated the present stability problem for the case $\beta = 0$. Their results imply that the $c_i = 0{\cdot}05$ curve should approach the neutral curve asymptotically as $k^2, \beta \to 0$.

waves of interest. If not, other amplified waves could be present. Although no proof can be offered, it does not seem likely that other amplified waves are present. The reason is that for $\beta = 0$ equation (3.4) reduces to a form of Legendre's equation. For this case it can be shown that no other neutral wave solutions exist and hence no additional amplified waves can be present. Since the effect of β is stabilizing it seems highly unlikely that for $\beta > 0$ any new types of unstable waves are present. Hence it appears that only one mode of unstable waves exists for the present case.

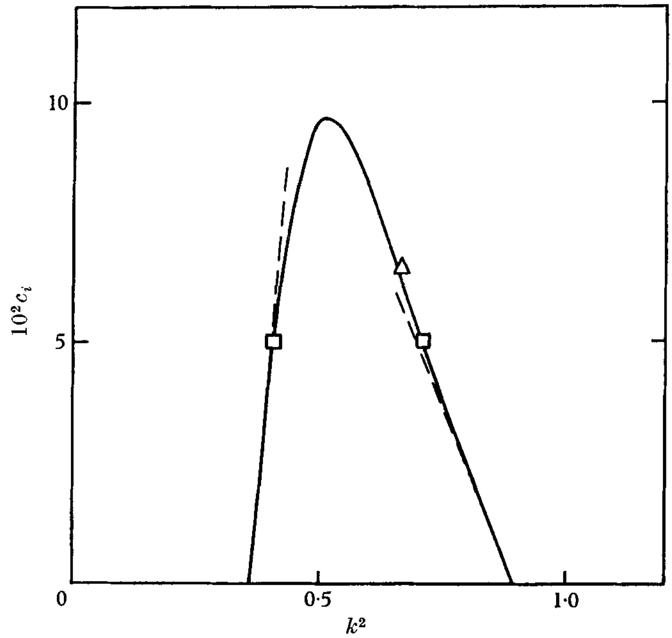

FIGURE 4. The estimated curve c_i versus k^2 for $\beta = 1/3^{\frac{1}{2}}$. The squares represent points from the $c_i = 0{\cdot}05$ curve in figure 3. The dashed lines are the slopes $\partial c_i/\partial k^2$ calculated from the neutral solutions. The triangle represents an estimate of c_i obtained from $\partial c_i/\partial \beta$ calculated along the neutral curve.

We now consider c_i as a function of k^2 for $\beta = 3^{-\frac{1}{2}}$ and $\beta = 2 \times 3^{-\frac{3}{2}}$. For $\beta = 3^{-\frac{1}{2}}$, the values of c_i can be estimated reasonably well and the plot of c_i versus k^2 is shown in figure 4. In this figure the dashed lines represent the slopes $\partial c_i/\partial k^2$ calculated at the neutral curve and the triangle has the same meaning as before. The squares represent points taken from the $c_i = 0{\cdot}05$ curve in figure 3. In figure 5 is shown the estimated curve for c_i as a function of k^2 for $\beta = 2 \times 3^{-\frac{3}{2}}$. It is evident that the values of c_i cannot be estimated as accurately as for the previous case; a reasonable guess for the maximum error is about 25 %.

5. Calculation of the amplification rates for the asymmetric current and the symmetric jet

In this section we apply the above theory to calculate the amplification rates for a hypothetical basic current in the upper atmosphere. These amplification rates will be then compared with those obtained when a similar calculation is made for a symmetric jet.

For the present case we consider a basic current of non-dimensional form $U = -\tanh y$ with the point $y = 0$ located at 35 degrees North latitude. The total velocity shear is taken to be $30\,\text{m}\,\text{sec}^{-1}$ so that $|U_0| = 15\,\text{m}\,\text{sec}^{-1}$. At 35 degrees $\beta_0 = 1{\cdot}875 \times 10^{-11}\,\text{m}^{-1}\,\text{sec}^{-1}$. Non-dimensional β is taken as $2 \times 3^{-\frac{3}{2}}$ so that the value of L is $5{\cdot}55 \times 10^5\,\text{m}$. These values of $|U_0|$ and L are of a reasonable magnitude to occur in the atmosphere.

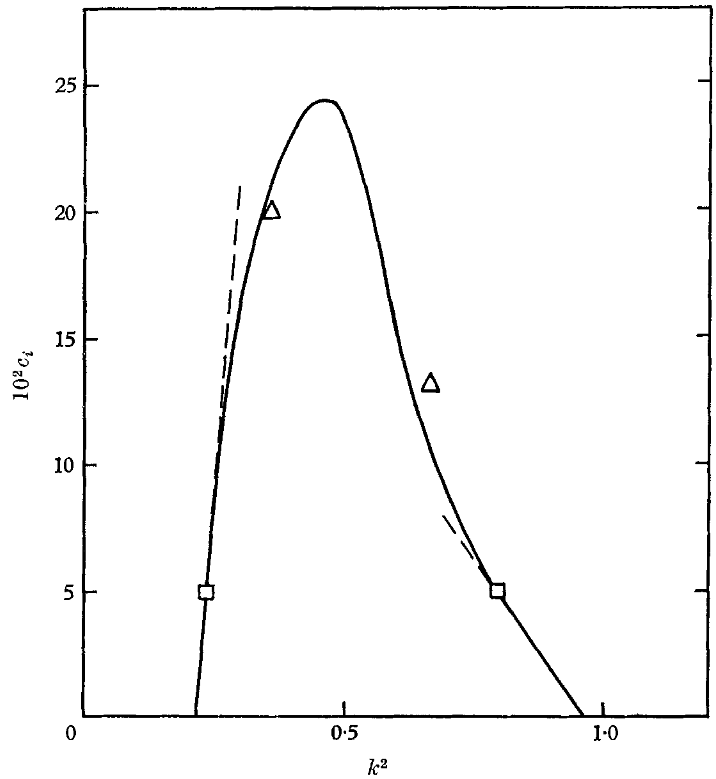

FIGURE 5. The estimated curve of c_i versus k^2 for $\beta = 2/3^{\frac{3}{2}}$. The notation is the same as in figure 4.

n	Amplification rate (days)
5	5·6
6	2·9
7	2·9
8	6·7
9	34·2

TABLE 2. Amplification rates for $U = -\tanh y$

In table 2 are shown the amplification rates for the unstable waves. In this table n is the number of waves around the globe at 35 degrees latitude and the amplification rates are given as the number of days it takes a disturbance to amplify by a factor of e. It is noted that the most unstable waves are $n = 6$ and $n = 7$ which have an e-fold amplification in 2·9 days. Since the data in this table are obtained from figure 5, the values of the amplification rates should be correct to within about 25%.

The above amplification rates are to be compared to those obtained for a symmetric jet (Lipps 1962, 1963). We consider a basic current of dimensional form $U = V \operatorname{sech}^2 y/L + V_0$ with the point $y = 0$ again located at 35 degrees latitude. The velocity shear is taken to be 30 m sec^{-1}. In order to find a value for the half-width L we define the non-dimensional number $B = \beta_0 L^2/V$ as in Lipps (1962). In the above case where $\beta = 2 \times 3^{-\frac{3}{2}}$ this value of β is one half the critical value of β for which the flow first becomes stable. If we use the same criterion here in choosing B we find $B = \frac{1}{3}$. With this value of B, the value of β_0 given above and $V = 30$ m sec^{-1} we find $L = 7{\cdot}31 \times 10^5$.

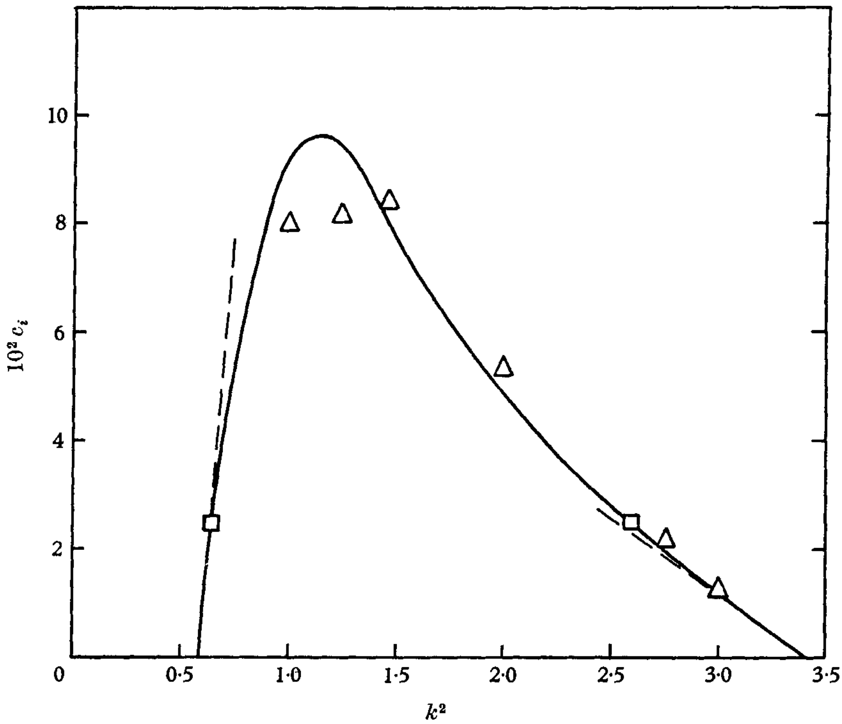

FIGURE 6. The estimated curve of c_i versus k^2 for the symmetric jet with $B = \frac{1}{3}$. The notation is explained in the text.

In figure 6 is shown the non-dimensional graph of c_i versus k^2 (k is again the wave-number) for the symmetric jet discussed above. In this figure the dashed lines are the slopes $\partial c_i/\partial k^2$ calculated from the neutral solutions and the squares represent points along the $c_i = 0{\cdot}025$ curve in figure 5 of Lipps (1963). The triangles represent estimates of c_i obtained from the derivatives $\partial c_i/\partial B$ along the neutral curve.

The amplification rates found in table 3 are obtained from the data in figure 6. It is noted that the most unstable wave is wave-number 8 which amplifies by a factor of e in 2·7 days. It is also seen that a broader band of wave-numbers are unstable for this basic flow. The maximum error in the magnitudes of the growth rates is again estimated to be about 25 %.

The amplification rates in table 3 are for the symmetric disturbances. In addition to these unstable waves the antisymmetric disturbances† are unstable for wave-numbers 1 and 2. However, the growth rates of these disturbances are very slow and therefore are not included in table 3.

n	Amplification rate (days)
6	7·8
7	3·2
8	2·7
9	3·1
10	4·1
11	5·7
12	9·4
13	51·3

TABLE 3. Amplification rates for the symmetric jet

6. Summary and conclusions

In this paper the barotropic stability of a zonal current of form $U = -\tanh y$ is studied. In § 2 the problem is formulated, and the boundary conditions and the basic equation to be satisfied by the perturbations are given. It is also shown in this section that the stability characteristics of $U = \tanh y$ and $U = -\tanh y$ are identical.

In § 3 the neutral waves are discussed, and the neutral curve in figure 3 is obtained. In § 4 the amplified waves are studied by means of the derivatives $\partial c/\partial s$ and $\partial c/\partial \beta$. From these derivatives the estimated curve for $c_i = 0{\cdot}05$ in figure 3 is found. It is noted that for this basic flow there is only one mode of unstable waves. For the symmetric jet Lipps (1962, 1963) there are two modes of unstable waves; namely, the symmetric and the antisymmetric disturbances.

The magnitudes of the growth rates for the asymmetric current and the symmetric jet are compared in § 5 by means of two examples. The first example is a $U = -\tanh y$ basic flow centred at 35 degrees latitude. The total shear is 30 m sec^{-1} and the half-width L is $5{\cdot}55 \times 10^5$ m. For this case wave-numbers 5–9 are unstable. The most unstable disturbances are wave-numbers 6 and 7 which have an e-fold amplification in 2·9 days.

The second example is a symmetric jet centred at 35 degrees with a total shear of 30 m sec^{-1} and a half-width of $7{\cdot}31 \times 10^5$ m. In this case wave-numbers 6–13 are unstable for the symmetric disturbances. The most unstable disturbance is wave-number 8 which has an e-fold amplification in 2·7 days. In addition wave-numbers 1 and 2 are unstable for the anti-symmetric disturbances, but the growth rates are much slower.

† It is to be noted that figure 4 in Lipps (1962) for the antisymmetric disturbances is in error. The correct form of this figure using the B notation is shown as figure 6 in Lipps (1963).

From a comparison of the two examples it is seen that the maximum growth rates are comparable. However, the symmetric jet appears to be unstable over a larger band of wave-numbers and the most unstable wavelength is somewhat shorter. (Wave-number 8 is 4100 km and wave-numbers 6 and 7 are 5500 and 4700 km respectively.)

In conclusion it is of interest to compare the growth rate of baroclinic disturbances with the amplification rates found above. Kuo (1952) calculates the growth rates for the unstable waves associated with a basic current which increases linearly the height. For a vertical shear of $3\,\mathrm{m\,sec^{-1}km^{-1}}$ (which is a shear of $30\,\mathrm{m\,sec^{-1}}$ in 10 km) Kuo finds that the most unstable disturbance has a wavelength of 4150 km and its growth rate corresponds to an e-fold amplification in 1·45 days. Thus it appears from the above examples that the wavelengths of the most unstable barotropic and baroclinic waves are roughly equivalent but that the baroclinic disturbances grow much faster than the barotropic disturbances.

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Appendix. The limiting values of $\partial c/\partial s$ and $\partial c/\partial \beta$ at $\beta = 0$, $k^2 = 0$

In order to calculate $\partial c/\partial s$ and $\partial c/\partial \beta$ at $\beta = 0$, $k^2 = 0$ we take the limit of (4.2) and (4.3) as we approach this point along the neutral curve. First it is noted that if we set $z = \tanh y$ equations (4.2) and (4.3) become

$$\frac{\partial c}{\partial s} = \int_{-1}^{1} \left(\frac{1-z}{1+z}\right)^{-c} dz \bigg/ \int_{-1}^{1} \frac{H}{-z-c}\left(\frac{1-z}{1+z}\right)^{-c} dz, \tag{A 1}$$

$$\frac{\partial c}{\partial \beta} = \int_{-1}^{1} \frac{1}{-z-c}\left(\frac{1-z}{1+z}\right)^{-c} dz \bigg/ \int_{-1}^{1} \frac{H}{-z-c}\left(\frac{1-z}{1+z}\right)^{-c} dz, \tag{A 2}$$

$$H = -2(1-z^2) - 2zc + 2c^2,$$

where we have substituted (3.8) for ϕ and have cancelled out one factor of $U-c$ in the denominators of (A 1) and (A 2) since $\beta - U'' = 0$ at the point where $U - c = 0$.

The derivative $\partial c/\partial s$ is considered first. The numerator of (A 1) can be put in the form

$$\int_{-1}^{1} \left(\frac{1-z}{1+z}\right)^{-c} dz = \int_{-1}^{-1+\epsilon} \left(\frac{1-z}{1+z}\right)^{-c} dz + \int_{-1+\epsilon}^{1} \left(\frac{1-z}{1+z}\right)^{-c} dz$$
$$= \left(\frac{2}{\epsilon}\right)^{-c} \frac{\epsilon}{1+c} + \sigma(\epsilon) + \int_{-1+\epsilon}^{1} \left(\frac{1-z}{1+z}\right)^{-c} dz, \tag{A 3}$$

where $\epsilon \ll 1$. The denominator of $\partial c/\partial s$ can be put in the form

$$\int_{-1}^{1} \frac{H}{-z-c}\left(\frac{1-z}{1+z}\right)^{-c} dz = \int_{-1}^{-1+\epsilon} \frac{H}{-z-c}\left(\frac{1-z}{1+z}\right)^{-c} dz + (6c^2-2)\left(\frac{1+c}{1-c}\right)^{-c} \pi i + g, \tag{A 4}$$

where the second term is the contribution from integrating around the singularity at $y = y_c$, and g is the remainder of the integral above the first two terms. The first term becomes

$$\int_{-1}^{-1+\epsilon} \frac{H}{-z-c}\left(\frac{1-z}{1+z}\right)^{-c} dz = \left(\frac{2}{\epsilon}\right)^{-c} \frac{2c}{1-c}\epsilon + \sigma(\epsilon). \tag{A 5}$$

The derivative $\partial c/\partial s$ is now found by taking the limit $c \to 1$. During this limiting process ϵ is held fixed as a very small, but finite quantity. In the limit the numerator becomes

$$\lim_{c\to-1} \int_{-1}^{1} \left(\frac{1-z}{1+z}\right)^{-c} dz = \lim_{c\to-1} \frac{2}{1+c} + f(\epsilon), \tag{A 6}$$

where $f(\epsilon)$ is a finite term which is a function of ϵ. The limit of the denominator becomes

$$\lim_{c\to-1} \int_{-1}^{1} \frac{H}{-z-c}\left(\frac{1-z}{1+z}\right)^{-c} dz = -2 + \lim_{c\to-1} 2(1+c)\,\pi i + \sigma(\epsilon). \tag{A 7}$$

In this expression we have used that $\lim_{c\to-1} g = \sigma(\epsilon)$ as can be verified directly. It now follows that the limiting value of $\partial c/\partial s$ becomes

$$\lim_{c\to-1} \frac{\partial c}{\partial s} = -\infty - \pi i. \tag{A 8}$$

The limiting value of $\partial c/\partial \beta$ can be found in a similar fashion. The limiting form of the numerator in (A 2) becomes

$$\lim_{c\to-1} \int_{-1}^{1} \frac{1}{-z-c}\left(\frac{1-z}{1+z}\right)^{-c} dz = \lim_{c\to-1} \left(\frac{1}{1+c} + \tfrac{1}{2}(1+c)\,\pi i\right) + f(\epsilon), \tag{A 9}$$

where f is again a finite real term which is a function of ϵ. In the limit $\partial c/\partial \beta$ becomes

$$\lim_{c\to-1} \frac{\partial c}{\partial \beta} = -\infty - \frac{\pi}{2} i. \tag{A 10}$$

The limiting values of $\partial c/\partial s$ and $\partial c/\partial \beta$ given by (A 8) and (A 10) are the values shown in table 1 for the point $\beta = 0$, $k^2 = 0$.

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